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APPLICATION OF LOCAL CO-ORDINATE SYSTEMS TO THE SOLUTION OF INTEGRAL EQUATIONS REFERRED TO PLANE FLUID FLOW PROBLEMS

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SUMMARY

Two-dimensional fluid flow problems expressed in terms of velocity potentials or stream functions are often summarized as boundary-value problems for the Laplace or Poisson equations, or the homogeneous or non-homogeneous biharmonic equations. Simple local co-ordinate systems have been applied to the solution of integral equations associated with these boundary-value problems. This procedure has been shown to be an efficient technique in the numerical solution of fluid flow problems.

KEY WORDS Fluid Flow Problems Integral Equations Local Co-ordinate Systems Wave Structure–Interaction Problems Wind-driven Current Problems

INTRODUCTION

Fluid flow problems are often solved by introducing a velocity potential or stream function. This procedure often leads to a boundary-value problem for the Laplace or Poisson equations or the homogeneous or non-homogeneous biharmonic equations. The solution of such boundary-value problems is, in general, difficult. Increase in memory size of digital computers have now made direct numerical solution of the boundary-value problems practical.

The boundary element method (BEM) has proved itself to be an efficient and accurate numerical scheme in the solution of fluid flow problems. An important step in solving boundary-value problems using this method is the solution of an integral equation. Analytical solutions to integral equations are generally not practical. A straightforward numerical approach is to replace the integral equation by a system of simultaneous linear equations. This is usually achieved under the assumption of uniformly distributed functions occurring in the integrands, over the boundary or internal elements. A review of this procedure can be found in References 1 and 2.

In this paper simple local co-ordinate systems are applied to the solution of integral equations arising from the BEM formulation for plane fluid flow problems. A two-dimensional region has been divided into triangular cells and the boundary of the region has been divided into line segments. The integral equations have been solved using linearly varying functions over the boundary elements or triangular cells. This type of approach has been applied to the solution of free surface flow problems. In order to verify the numerical results, computational examples have been chosen in which either the analytical solution is known or experimental data are available.

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SOLVING INTEGRAL EQUATIONS USING LOCAL CO-ORDINATE SYSTEMS

Integral equation referred to plane fluid flow problems

The description of plane fluid flow problems as boundary-value problems often leads to the Laplace or Poisson equations or the homogeneous or non-homogeneous biharmonic equations with proper boundary conditions. There are some formulations of the BEM which make the solution of such boundary-value problems possible.[1-6]

In the present paper the formulations based on Green's second identity and the Rayleigh–Green identity are used. Accordingly, the solutions of the following two integral equations are considered:

$$\int_{\hat{R}_1} (\hat{U}_1 \nabla^2 \hat{V}_1 - \hat{V}_1 \nabla^2 \hat{U}_1)\, d\hat{R}_1 = \int_{\hat{S}_1} \left(\hat{U}_1 \frac{\partial \hat{V}_1}{\partial n} - \hat{V}_1 \frac{\partial \hat{U}_1}{\partial n} \right) d\hat{S}_1, \tag{1}$$

$$\int_{\hat{R}_2} (\hat{U}_2 \nabla^4 \hat{V}_2 - \hat{V}_2 \nabla^4 \hat{U}_2)\, d\hat{R}_2 = \int_{\hat{S}_2} \left[\hat{U}_2 \frac{\partial(\nabla^2 \hat{V}_2)}{\partial n} - \nabla^2 \hat{V}_2 \frac{\partial \hat{U}_2}{\partial n} + \nabla^2 \hat{U}_2 \frac{\partial \hat{V}_2}{\partial n} - \hat{V}_2 \frac{\partial(\nabla^2 \hat{U}_2)}{\partial n} \right] d\hat{S}_2, \tag{2}$$

where it is assumed that the domain $\hat{R}_1(\hat{R}_2)$, its boundary $\hat{S}_1(\hat{S}_2)$ and the functions $\hat{U}_1$, $\hat{V}_1(\hat{U}_2, \hat{V}_2)$ satisfy the condition for which Green's second identity (the Rayleigh–Green identity) is valid.[1,7-10]

Making use of equation (1) and of the fundamental solution, the solution in the domain $\hat{R}$ bounded by $\hat{S}$ of the boundary-value problem for the Laplace equation ($\nabla^2 U = 0$) or the Poisson equation ($\nabla^2 U = \hat{p}$) requires a numerical solution of the following integral equation:

$$\int_{\hat{S}} \left[U \frac{\partial(\ln r)}{\partial n} - \ln r \frac{\partial U}{\partial n} \right] d\hat{S} + \int_{\hat{R}} \ln r \nabla^2 U\, d\hat{R} = 0, \tag{3}$$

where r is the distance from an arbitrary point, P, to a point, Q, on the boundary $\hat{S}$; $\mathbf{n}$ is the unit outward normal to the boundary.

In the case of the biharmonic homogeneous equation ($\nabla^4 U = 0$) or the biharmonic non-homogeneous equation ($\nabla^4 U = \hat{q}$) the following integral equations should be solved:

$$\int_{\hat{S}} \left[\nabla^2 U \frac{\partial(\ln r)}{\partial n} - \ln r \frac{\partial(\nabla^2 U)}{\partial n} \right] d\hat{S} + \int_{\hat{R}} \ln r \nabla^4 U\, d\hat{R} = 0, \tag{4}$$

$$\int_{\hat{S}} \left\{ U \frac{\partial[\nabla^2(r^2 \ln r)]}{\partial n} - \nabla^2(r^2 \ln r) \frac{\partial U}{\partial n} + \nabla^2 U \frac{\partial(r^2 \ln r)}{\partial n} - r^2 \ln r \frac{\partial(\nabla^2 U)}{\partial n} \right\} d\hat{S} + \int_{\hat{R}} r^2 \ln r \nabla^4 U\, d\hat{R} = 0 \tag{5}$$

Solutions of the integral equation for the Laplace and Poisson equations

The first step in the numerical solution of the boundary-value problem for the Laplace equation formulated in terms of an integral equation is the subdivision of the boundary $\hat{S}$ into N suitably small straight-line segments.[3,11] Then equation (3) becomes

$$\sum_{j=1}^{N} \int_{\hat{S}_j} \left[U \frac{\partial}{\partial n}(\ln r) - \ln r \frac{\partial U}{\partial n} \right] d\hat{S}_j = 0. \tag{6}$$

If it is assumed that

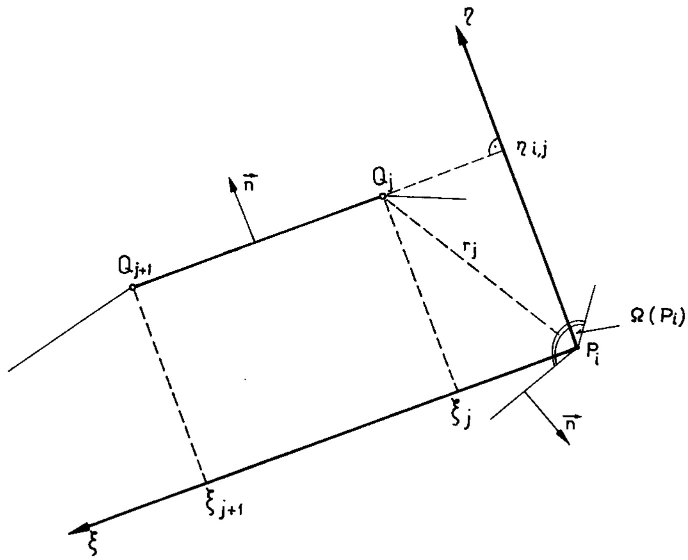

Figure 1. ξ–η co-ordinate system

1. the point P_i is placed at the beginning of the ith segment
2. the origin of the local co-ordinate system (ξ, η) is located at the point P_i (Figure 1)
3. between a pair of node points Q_j, Q_{j+1} (Figure 1) the function U and its outward normal derivative vary linearly according to

$$U = \frac{(U)_{j+1} - (U)_j}{\xi_{j+1} - \xi_j}\xi + \frac{\xi_{j+1}(U)_j - \xi_j(U)_{j+1}}{\xi_{j+1} - \xi_j} = A_{11}\xi + A_{12}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1}, \tag{7}$$

$$\frac{\partial U}{\partial n} = \frac{\left(\frac{\partial U}{\partial n}\right)_{j+1} - \left(\frac{\partial U}{\partial n}\right)_j}{\xi_{j+1} - \xi_j}\xi + \frac{\xi_{j+1}\left(\frac{\partial U}{\partial n}\right)_j - \xi_j\left(\frac{\partial U}{\partial n}\right)_{j+1}}{\xi_{j+1} - \xi_j} = A_{21}\xi + A_{22}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1} \tag{8}$$

then the substitution of equations (7) and (8) into equation (6) yields an algebraic equation of the following form:

$$\sum_{j=1}^{N} \int_{\xi_j}^{\xi_{j+1}} \left[U\frac{\partial}{\partial n}(\ln r) - \frac{\partial U}{\partial n}\ln r \right] d\xi = \sum_{j=1}^{N} (I^{\mathrm{I}}_{i,j} + I^{\mathrm{II}}_{i,j}). \tag{9}$$

The quantities $I^{\mathrm{I}}_{i,j}$ and $I^{\mathrm{II}}_{i,j}$ can be written as

$$I^{\mathrm{I}}_{i,j} = \left[\tfrac{1}{2}A_{11}\eta_{i,j}\ln(\eta^2_{i,j} + \xi^2) + A_{12}\arctan\left(\frac{\xi}{\eta_{i,j}}\right)\right]\Bigg|_{\xi_j}^{\xi_{j+1}}, \quad j \neq i-1, j \neq i, \tag{10}$$

$$I^{\mathrm{I}}_{i,i-1} + I^{\mathrm{I}}_{i,i} = -\Omega(P_i)(U)_i, \tag{11}$$

$$I^{\mathrm{II}}_{i,j} = \Bigg\{ -\frac{A_{21}}{4}(\eta^2_{i,j} + \xi^2)[\ln(\eta^2_{i,j} + \xi^2) - 1] - \frac{A_{22}}{2}\Bigg[\xi\ln(\eta^2_{i,j} + \xi^2) + 2\eta_{i,j}\arctan\left(\frac{\xi}{\eta_{i,j}}\right) - 2\xi\Bigg]\Bigg\}\Bigg|_{\xi_j}^{\xi_{j+1}}, \tag{12}$$

where $\Omega(P_i)$ is the inner angle relative to $\hat{R}$, between $\overline{Q_iQ_{i-1}}$ and $\overline{Q_iQ_{i+1}}$.

In the case of the Poisson equation, if a particular solution of this equation is not known, it is

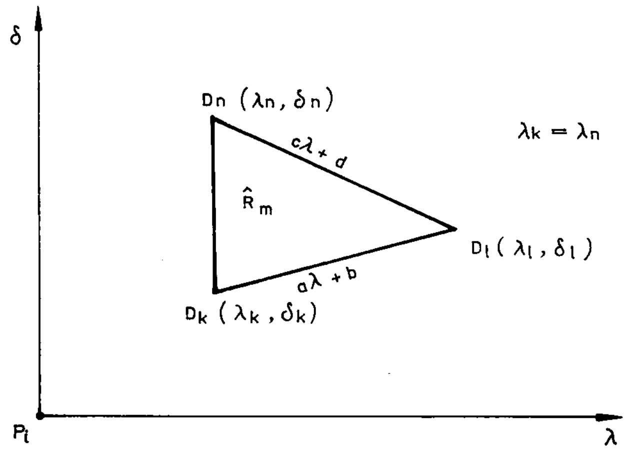

Figure 2. λ–δ co-ordinate system

necessary to discretize the surface $\hat{S}$, and the domain $\hat{R}$, as well. The boundary $\hat{S}$ is divided into N small segments and the domain $\hat{R}$ is divided into M small triangular cells. The variable $\hat{p}$ for such cells is approximated by a linear function. Then equation (3) becomes

$$\sum_{j=1}^{N}\int_{\hat{S}_j}\left[U\frac{\partial}{\partial n}(\ln r)-\ln r\frac{\partial U}{\partial n}\right]\mathrm{d}\hat{S}_j+\sum_{m=1}^{M}\int_{\hat{R}_m}\hat{p}\ln r\,\mathrm{d}\hat{R}_m=0. \tag{13}$$

The procedure continues by assuming that

1. The point P_i is placed at the beginning of ith segment.
2. The origins of the local co-ordinate systems (ξ,η) and (λ,δ) are located at the point P_i (Figures 1 and 2).
3. Between a pair of node points Q_j, Q_{j+1} (Figure 1) the function U and its outward normal derivative vary linearly. For each triangular cell $\hat{R}_m (m=1,2,\ldots,M)$ the function $\hat{p}$ is approximated by the following linear function (Figure 2):

$$\hat{p}=A_{31}\delta+A_{32}\lambda+A_{33}\quad\begin{cases}a\lambda+b\leqslant\delta\leqslant c\lambda+d,\\ \lambda_k\leqslant\lambda\leqslant\lambda_l,\end{cases} \tag{14}$$

where

$$A_{31}=-[\hat{p}_k(\lambda_l-\lambda_n)+\hat{p}_n(\lambda_k-\lambda_l)]/[(\delta_k-\delta_n)(\lambda_k-\lambda_l)], \tag{15}$$

$$A_{32}=-[-\hat{p}_k(\delta_l-\delta_n)+\hat{p}_l(\delta_k-\delta_n)-\hat{p}_n(\delta_k-\delta_l)]/[(\delta_k-\delta_n)(\lambda_k-\lambda_l)], \tag{16}$$

$$A_{33}=-[\hat{p}_k(\delta_l\lambda_n-\delta_n\lambda_l)+\hat{p}_l(\delta_n\lambda_k-\delta_k\lambda_n)+\hat{p}_n(\delta_k\lambda_l-\delta_l\lambda_k)]/[(\delta_k-\delta_n)(\lambda_k-\lambda_l)]. \tag{17}$$

Making use of the above assumption the following algebraic equation is obtained from equation (13):

$$\sum_{j=1}^{N}\int_{\xi_j}^{\xi_{j+1}}\left[U\frac{\partial}{\partial n}(\ln r)-\frac{\partial U}{\partial n}\ln r\right]\mathrm{d}\xi+\sum_{m=1}^{M}\int_{\lambda_k}^{\lambda_l}\int_{a\lambda+b}^{c\lambda+d}(A_{31}\delta+A_{32}\lambda+A_{33})\,\mathrm{d}\delta\,\mathrm{d}\lambda$$
$$=\sum_{j=1}^{N}(I^{\mathrm{I}}_{i,j}+I^{\mathrm{II}}_{i,j})+\sum_{m=1}^{M}(I^{\mathrm{III}}_{i,m}+I^{\mathrm{IV}}_{i,m}+I^{\mathrm{V}}_{i,m})=0, \tag{18}$$

where $I^{\mathrm{III}}_{i,m}$, $I^{\mathrm{IV}}_{i,m}$ and $I^{\mathrm{V}}_{i,m}$ are defined in Appendix I.

Applying equations (9) or (18) at each point $P_i (i=1,2,\ldots,N)$, in a well-posed problem, N

simultaneous equations are obtained in an equal number of unknowns. The solution of these equations provides the boundary data that can be used in equation (3) to find the solution at any interior point.

Solution of the integral equation for the homogeneous and non-homogeneous biharmonic equations

The boundary $\hat{S}$ is subdivided as in the previous discussion. Then in the case of a biharmonic homogeneous equation, equations (4) and (5) yield

$$\sum_{j=1}^{N} \int_{\hat{S}_j} \left[\nabla^2 U \frac{\partial}{\partial n}(\ln r) - \ln r \frac{\partial}{\partial n}(\nabla^2 U) \right] d\hat{S}_j = 0, \tag{19}$$

$$\sum_{j=1}^{N} \int_{\hat{S}_j} \left\{ U \frac{\partial}{\partial n}[\nabla^2 (r^2 \ln r)] - \nabla^2 (r^2 \ln r) \frac{\partial U}{\partial n} + \nabla^2 U \frac{\partial}{\partial n}(r^2 \ln r) - r^2 \ln r \frac{\partial(\nabla^2 U)}{\partial n} \right\} d\hat{S}_j = 0. \tag{20}$$

Assume that the point P_i and the local co-ordinate system (ξ, η) are referenced to the beginning of the ith segment and that between a pair of node points Q_j, Q_{j+1} the functions U, $\partial U/\partial n$, $\nabla^2 U$, $\partial(\nabla^2 U)/\partial n$ vary linearly as

$$\begin{aligned} U &= \frac{(U)_{j+1} - (U)_j}{\xi_{j+1} - \xi_j} \xi + \frac{\xi_{j+1}(U)_j - \xi_j (U)_{j+1}}{\xi_{j+1} - \xi_j} \\ &= B_{11}\xi + B_{12}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1}, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial U}{\partial n} &= \frac{\left(\frac{\partial U}{\partial n}\right)_{j+1} - \left(\frac{\partial U}{\partial n}\right)_j}{\xi_{j+1} - \xi_j} \xi + \frac{\xi_{j+1}\left(\frac{\partial U}{\partial n}\right)_j - \xi_j \left(\frac{\partial U}{\partial n}\right)_{j+1}}{\xi_{j+1} - \xi_j} \\ &= B_{21}\xi + B_{22}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1}, \end{aligned} \tag{22}$$

$$\begin{aligned} \nabla^2 U &= \frac{(\nabla^2 U)_{j+1} - (\nabla^2 U)_j}{\xi_{j+1} - \xi_j} \xi + \frac{\xi_{j+1}(\nabla^2 U)_j - \xi_j (\nabla^2 U)_{j+1}}{\xi_{j+1} - \xi_j} \\ &= B_{31}\xi + B_{32}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1}, \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial(\nabla^2 U)}{\partial n} &= \frac{\left[\frac{\partial(\nabla^2 U)}{\partial n}\right]_{j+1} - \left[\frac{\partial(\nabla^2 U)}{\partial n}\right]_j}{\xi_{j+1} - \xi_j} \xi + \frac{\xi_{j+1}\left[\frac{\partial(\nabla^2 U)}{\partial n}\right]_{j+1} - \xi_j \left[\frac{\partial(\nabla^2 U)}{\partial n}\right]_{j+1}}{\xi_{j+1} - \xi_j} \\ &= B_{41}\xi + B_{42}, \quad \xi_j \leqslant \xi \leqslant \xi_{j+1}. \end{aligned} \tag{24}$$

Then the two following algebraic equations are obtained from equations (19) and (20):

$$\sum_{j=1}^{N} \int_{\xi_j}^{\xi_{j+1}} \left[\nabla^2 U \frac{\partial}{\partial n}(\ln r) - \ln r \frac{\partial}{\partial n}(\nabla^2 U) \right] d\xi = \sum_{j=1}^{N} (\tilde{I}^{\mathrm{I}}_{i,j} + \tilde{I}^{\mathrm{II}}_{i,j}) = 0, \tag{25}$$

$$\begin{aligned} \sum_{j=1}^{N} \int_{\xi_j}^{\xi_{j+1}} \Big\{ & U \frac{\partial}{\partial n}[\nabla^2 (r^2 \ln r)] - \nabla^2 (r^2 \ln r) \frac{\partial U}{\partial n} + \nabla^2 U \frac{\partial}{\partial n}(r^2 \ln r) \\ & - r^2 \ln r \frac{\partial}{\partial n}(\nabla^2 U) \Big\} d\xi = \sum_{j=1}^{N} (\mathrm{II}^{\mathrm{I}}_{i,j} + \mathrm{II}^{\mathrm{II}}_{i,j} + \mathrm{II}^{\mathrm{III}}_{i,j} + \mathrm{II}^{\mathrm{IV}}_{i,j}) = 0. \end{aligned} \tag{26}$$

The quantities $\tilde{I}^{\text{I}}_{i,j}$ and $\tilde{I}^{\text{II}}_{i,j}$ can be calculated from equations (10)–(12) using B_{31}, B_{32}, B_{41} and B_{42} as alternatives to A_{11}, A_{12}, A_{21} and A_{22}, respectively. The remaining unknown quantities are calculated from the following formulae:

$$\text{II}^{\text{I}}_{i,j} = \left[2B_{11}\eta_{i,j}\ln(\eta^2_{i,j}+\xi^2) + 4B_{12}\arctan\left(\frac{\xi}{\eta_{i,j}}\right)\right]\Bigg|^{\xi_{j+1}}_{\xi_j}, \quad j \neq i-1, j \neq i, \tag{27}$$

$$\text{II}^{\text{I}}_{i,i-1} + \text{II}^{\text{I}}_{i,i} = -4\Omega(\text{P}_i)(U)_i, \tag{28}$$

$$\text{II}^{\text{II}}_{i,j} = \Bigg\{ B_{21}[-(\eta^2_{i,j}+\xi^2)\ln(\eta^2_{i,j}+\xi^2) + \eta^2_{i,j} - \xi^2] + B_{22}\Bigg[-2\xi\ln(\eta^2_{i,j}+\xi^2)$$
$$- 4\eta_{i,j}\arctan\left(\frac{\xi}{\eta_{i,j}}\right)\Bigg]\Bigg\}\Bigg|^{\xi_{j+1}}_{\xi_j}, \tag{29}$$

$$\text{II}^{\text{III}}_{i,j} = \Bigg\{ \tfrac{1}{2}B_{31}\eta_{i,j}[(\eta^2_{i,j}+\xi^2)\ln(\eta^2_{i,j}+\xi^2) - \eta^2_{i,j}] + B_{32}\eta_{i,j}\Bigg[\xi\ln(\eta^2_{i,j}+\xi^2)$$
$$+ 2\eta_{i,j}\arctan\left(\frac{\xi}{\eta_{i,j}}\right) - \xi\Bigg]\Bigg\}\Bigg|^{\xi_{j+1}}_{\xi_j}, \tag{30}$$

$$\text{II}^{\text{IV}}_{i,j} = \Bigg\{ \tfrac{1}{8}B_{41}(\eta^2_{i,j}+\xi^2)^2[-\ln(\eta^2_{i,j}+\xi^2) + \tfrac{1}{2}] + B_{42}\Bigg[-\tfrac{1}{2}\eta^2_{i,j}\xi\ln(\eta^2_{i,j}+\xi^2)$$
$$- \tfrac{1}{6}\xi^3\ln(\eta^2_{i,j}+\xi^2) - \tfrac{2}{3}\eta^3_{i,j}\arctan\left(\frac{\xi}{\eta_{i,j}}\right) + \tfrac{2}{3}\eta^2_{i,j}\xi + \tfrac{1}{9}\xi^3\Bigg]\Bigg\}\Bigg|^{\xi_{j+1}}_{\xi_j}. \tag{31}$$

If a particular solution to the biharmonic non-homogeneous equation is not known, the boundary $\hat{S}$ and the domain $\hat{R}$ are discretized as with the Poisson equation. The local co-ordinate systems (ξ, η) and (λ, δ) are used and it is assumed that

$$\hat{q} = B_{51}\delta + B_{52}\lambda + B_{53}\begin{cases} a\lambda + b \leqslant \delta \leqslant c\lambda + d, \\ \lambda_k \leqslant \lambda \leqslant \lambda_l, \end{cases} \tag{32}$$

where B_{51}, B_{52} and B_{53} are calculated from equations (15)–(17) using $\hat{q}_k, \hat{q}_l$ and $\hat{q}_n$ instead of $\hat{p}_k, \hat{p}_l$ and $(\hat{p}_n)$ respectively. Substitution of equations (21)–(24) and (32) into equations (4) and (5) yields the following two algebraic equations:

$$\sum_{j=1}^{N}(\tilde{I}^{\text{I}}_{i,j} + \tilde{I}^{\text{II}}_{i,j}) + \sum_{m=1}^{N}(\tilde{I}^{\text{III}}_{i,m} + \tilde{I}^{\text{IV}}_{i,m} + \tilde{I}^{\text{V}}_{i,m}) = 0, \tag{33}$$

$$\sum_{j=1}^{N}(\text{II}^{\text{I}}_{i,j} + \text{II}^{\text{II}}_{i,j} + \text{II}^{\text{III}}_{i,j} + \text{II}^{\text{IV}}_{i,j}) + \sum_{m=1}^{M}(\text{II}^{\text{V}}_{i,m} + \text{II}^{\text{VI}}_{i,m} + \text{II}^{\text{VII}}_{i,m}) = 0. \tag{34}$$

The quantities $\tilde{I}^{\text{III}}_{i,m}$, $\tilde{I}^{\text{IV}}_{i,m}$ and $\tilde{I}^{\text{V}}_{i,m}$ can be calculated from equations (68)–(70) using B_{51}, B_{52} and B_{53} instead of A_{31}, A_{32} and A_{33}, respectively. The remaining unknown quantities are defined in Appendix II.

Applying equations (25) and (26) or (33) and (34) at each point $\text{P}_i (i = 1, 2, \ldots, N)$, $2N$ simultaneous equations are obtained for an equal number of unknowns in well-posed problems. The solution of these equations provides the boundary data that can be used in equations (4) and (5) to find the solution at any interior point.

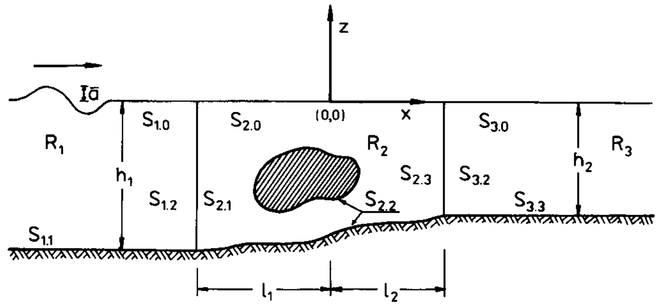

Figure 3. Definition sketch and co-ordinate system

COMPUTATIONAL EXAMPLES

Wave forces on impermeable objects

The situation considered for analysis is shown schematically in Figure 3. Additionally it is assumed that

1. The fluid is inviscid and incompressible.
2. The sea bottom is impervious.
3. A train of simple harmonic waves of frequency ω and small amplitude $\bar{a}$ is approaching the object from the left.
4. The velocity in each flow domain has a potential.

According to these assumptions the wave field can be specified by a velocity potential of the form

$$\Phi_l(x, z, t) = \mathrm{Re}\,[\phi_l(x, z)\mathrm{e}^{-\mathrm{i}\omega t}], \quad l = 1, 2, 3 \tag{35}$$

where Re denotes the real part and $\mathrm{i} = \sqrt{(-1)}$.

The wave field is completely specified if $\phi_l(x, z)$ is known. The boundary value problem for $\phi_l(x, z)$ may be written as follows:

$$R_l\colon \quad \nabla^2\phi_l = 0, \quad l = 1, 2, 3, \tag{36}$$

$$S_{l,0}\colon \quad \frac{\partial \phi_l}{\partial z} - \frac{\omega^2}{g}\phi_l = 0, \quad z = 0, \tag{37}$$

$$S_{l,l}\colon \quad \frac{\partial \phi_l}{\partial n_{l,l}} = 0, \tag{38}$$

$$S_{l,m} \text{ and } S_{m,l}\colon \quad \phi_l = \phi_m, \tag{39}$$

$$\frac{\partial \phi_l}{\partial n_{l,m}} = -\frac{\partial \phi_m}{\partial n_{m,l}}, \tag{40}$$

$$x \to -\infty\colon \quad \frac{\partial \tilde{\phi}_1}{\partial x} + \mathrm{i}k_1\tilde{\phi}_1 = 0, \tag{41}$$

$$x \to \infty\colon \quad \frac{\partial \phi_3}{\partial x} - \mathrm{i}k_2\phi_3 = 0, \tag{42}$$

where $S_{l,0}(S_{l,l})$ is the part of the R_l boundary domain that is a free water surface (impermeable part of the boundary), $S_{l,m}$ and $S_{m,l}$ are the common boundaries of the domains R_l and R_m, $\partial\phi_l/\partial n_{l,m}$ is the outward normal derivative of ϕ_l at $S_{l,m}$, and k_1 and k_2 are the wave numbers corresponding to the fluid depth in the domains R_1 and R_3.

The boundary value problem, equations (36)–(42), has been solved by applying the boundary element method in the domain R_2 with the following analytical solutions in the semi-infinite domains R_1 and R_3[12]:

$x \leqslant -l_1$:

$$\phi_1 = -\frac{ig\bar{a}}{\omega}e^{ik_1(x+l_1)}\frac{\cosh k_1(z+h_1)}{\cosh k_1h_1} + \tilde{\phi}_1, \tag{43}$$

$$\tilde{\phi}_1 = -\frac{ig\bar{R}}{\omega}e^{-ik_1(x+l_1)}\frac{\cosh k_1(z+h_1)}{\cosh k_1h_1} + \sum_{\alpha_n} -\frac{ig\bar{R}_{\alpha_n}}{\omega}e^{\alpha_n(x+l_1)}\frac{\cos\alpha_n(z+h_1)}{\cos\alpha_nh_1}; \tag{44}$$

$x \geqslant l_2$:

$$\phi_3 = \frac{-ig\bar{T}}{\omega}e^{ik_2(x-l_2)}\frac{\cosh k_2(z+h_2)}{\cosh k_2h_2} + \sum_{\beta_n} -\frac{ig\bar{T}_{\beta_n}}{\omega}e^{\beta_n(l_2-x)}\frac{\cos\beta_n(z+h_2)}{\cos\beta_nh_2}, \tag{45}$$

where $\bar{R}(\bar{T})$ is the amplitude of the reflected (transmitted) wave, $\bar{R}_{\alpha_n}(\bar{T}_{\beta_n})$ is the amplitude of the local standing wave and g is the acceleration due to gravity.

The eigenvalues $k_1, k_2, \alpha_n, \beta_n$ are roots of the equations

$$\frac{\omega^2}{g} = k_1\tanh(k_1h_1) = k_2\tanh(k_2h_2), \tag{46}$$

$$\frac{\omega^2}{g} = -\alpha_n\tan(\alpha_nh_1) = -\beta_n\tan(\beta_nh_2). \tag{47}$$

The forces on the object are determined by integrating the pressure acting on the surface of the object. The pressure, p, is expressed as[13,14]

$$p = -\rho\frac{\partial\Phi}{\partial t} - \frac{\rho}{2}(u^2+v^2), \tag{48}$$

where ρ is the fluid mass density and $u(v)$ is the velocity in the $x(z)$ direction. Although the solution for the velocity potential is obtained on the basis of linearity, in the calculation of pressure the squared terms of velocities are retained as a numerical experiment.

The method described above has been used to calculate wave forces on horizontal circular pipelines of radius R. The results are presented using the force coefficients defined below:

$$f_{2xm} = \frac{\text{maximum horizontal force per unit length of the pipe}}{\rho g R^2},$$

$$f_{2zm}^{+} = \frac{\text{maximum positive vertical force per unit length of the pipe}}{\rho g R^2},$$

$$f_{2zm}^{-} = \frac{\text{maximum negative vertical force per unit length of the pipe}}{\rho g R^2}.$$

In Figures 4 and 5, the numerical and experimental[14,15] values of the force coefficients are plotted ($k_1 = k_2 = k, h_1 = h_2 = h$). The predicted results are in reasonable agreement with the experimental data.

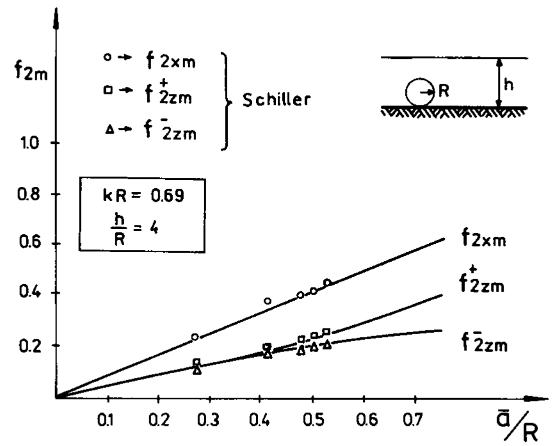

Figure 4. Variation of maximum force coefficient with relative amplitude

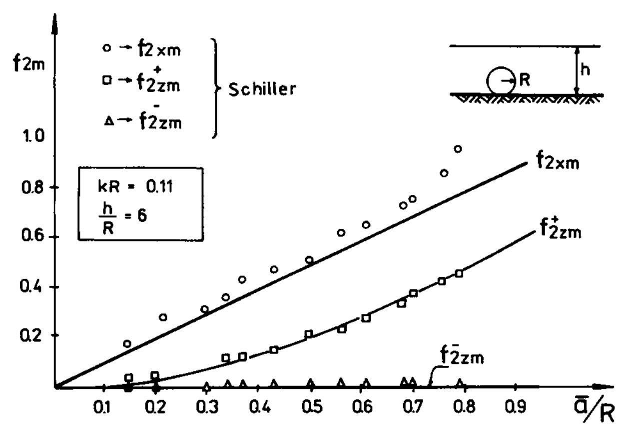

Figure 5. Variation of maximum force coefficient with relative amplitude

Wave reflection and transmission at rubble mound breakwater of arbitrary cross-section

Let us assume that water wave flow in a porous domain $\hat{R}$ is governed by the following equations:[16,17]

$$S\frac{\partial \Phi}{\partial t}+\frac{1}{\rho}(p+\gamma z)+f\omega\Phi=0, \tag{49}$$

$$\nabla^2\Phi=0, \tag{50}$$

where

$$f=\frac{1}{\omega}\frac{\displaystyle\int_{\hat{R}}\mathrm{d}\hat{R}\int_t^{t+\bar{T}_1}\varepsilon^2\left(\frac{\nu\mathbf{V}^2}{K}+\frac{C_f\varepsilon}{K^{1/2}}|\mathbf{V}|^3\right)\mathrm{d}t}{\displaystyle\int_{\hat{R}}\mathrm{d}\hat{R}\int_t^{t+\bar{T}_1}\varepsilon\mathbf{V}^2\,\mathrm{d}t} \tag{51}$$

is the damping coefficient, S is the inertial coefficient, γ is the weight density of fluid, ν is the kinematic viscosity, ε is the porosity, K is the intrinsic permeability, C_f is the turbulent damping coefficient, $\bar{T}_1$ is the wave period $\mathbf{V}=\nabla\Phi$.

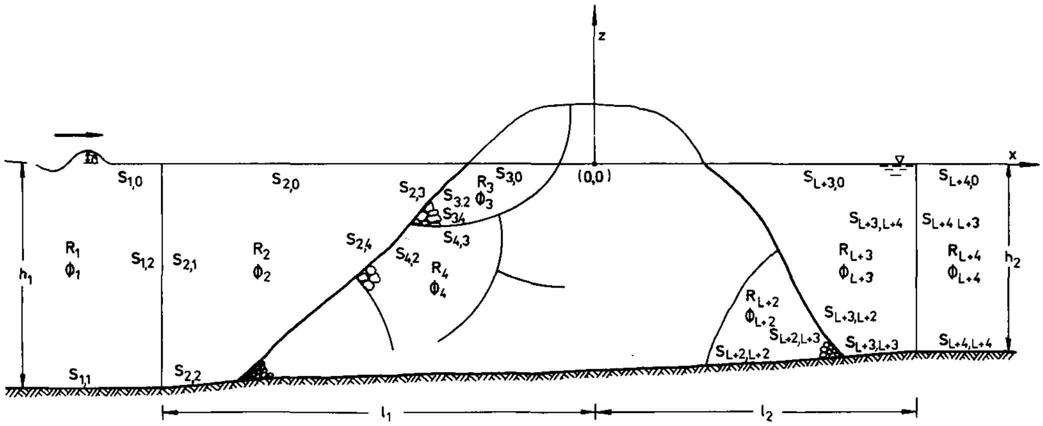

Figure 6. Definition sketch and co-ordinate system

The situation considered for analysis of the problem of wave reflection and transmission at a rubble mound breakwater of arbitrary cross-section is shown schematically in Figure 6.

It is assumed that the rubble mound breakwater is built from L porous layers ($l = 3, 4, \ldots, L+2$—Figure 6) each of known physical (ε_l) and hydraulic (K_l, C_{fl}) properties. Additionally it is assumed that the water wave flow in each porous layer is governed by equations (49) and (50) and that the assumptions taken into consideration in the problem of wave forces on objects are valid. Then, the harmonic velocity potentials may be expressed as the real part of a complex function in the form

$$\Phi_l(x,z,t) = \mathrm{Re}\,[\phi_l(x,z)\mathrm{e}^{-\mathrm{i}\omega t}], \quad l = 1,2,\ldots,L+4. \tag{52}$$

The boundary value problem for the spatial velocity potential, $\phi_l(x,z)$, can be written as follows:

$$R_l: \quad \nabla^2\phi_l(x,z) = 0, \quad l = 1,2,\ldots,L+4, \tag{53}$$

$$S_{l,0}: \quad \frac{\partial\phi_l}{\partial z} - \frac{\omega^2}{g}(S_l + \mathrm{i}f_l)\phi_l = 0, \quad z = 0, \tag{54}$$

$$S_{l,l}: \quad \frac{\partial\phi_l}{\partial n_{l,l}} = 0, \tag{55}$$

$$S_{l,m} \text{ and } S_{m,l}: \quad \begin{cases} (S_l + \mathrm{i}f_l)\phi_l = (S_m + \mathrm{i}f_m)\phi_m, & (56) \\ \varepsilon_l \dfrac{\partial\phi_l}{\partial n_{l,m}} = -\varepsilon_m \dfrac{\partial\phi_m}{\partial n_{m,l}}, & (57) \end{cases}$$

$$x \to -\infty: \quad \frac{\partial\tilde{\phi}_1}{\partial x} + \mathrm{i}k_1\tilde{\phi}_1 = 0, \tag{58}$$

$$x \to \infty: \quad \frac{\partial\phi_{L+4}}{\partial x} - \mathrm{i}k_2\phi_{L+4} = 0, \tag{59}$$

where $f_l = 0, S_l = 1$ and $\varepsilon_l = 1$ for $l = 1, 2, L+3, L+4$, and S_l is the inertial coefficient for the domain R_l (see equation (49)).

The boundary-value problem is solved iteratively by applying the boundary element method in the breakwater body and in the vicinity of the breakwater and the analytical solution in the exterior semi-infinite regions R_1 and R_{L+4}.

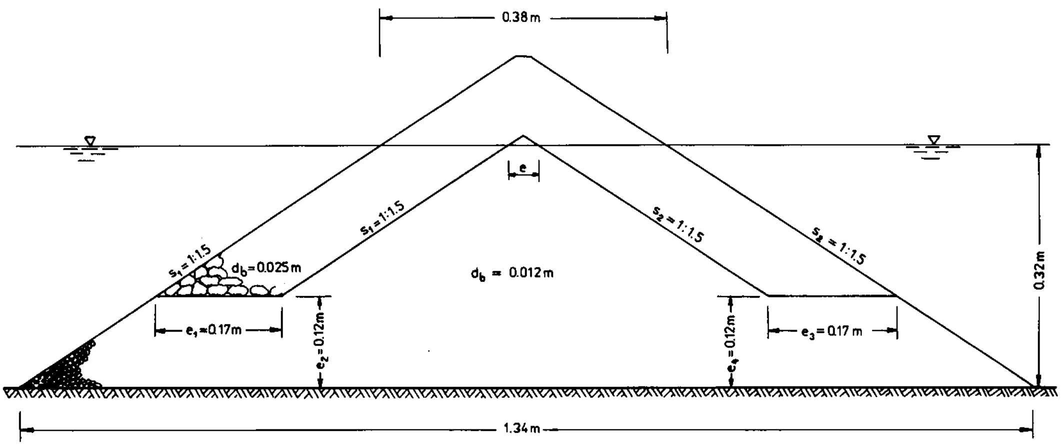

Figure 7. Trapezoidal layered breakwater (TW-2)

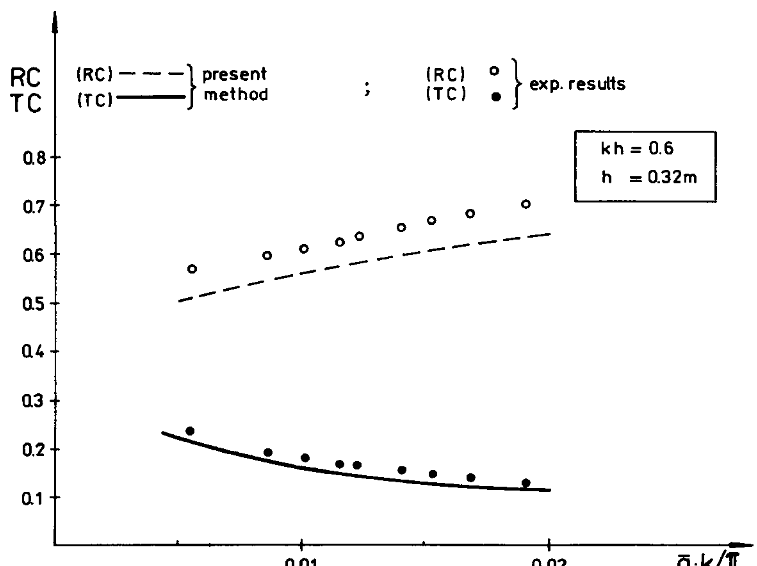

Figure 8. Reflection and transmission coefficients for trapezoidal layered breakwater (TW-2): dependence on wave steepness

Reflection and transmission coefficients for the two-layered trapezoidal-shaped structure shown in Figure 7 were computed as an example. The media properties are given by Sulisz.[17,18] The inertial coefficient, S, is unknown, and it is taken as equal to 1 by default. The reflection $(\mathrm{RC} = |\bar{R}|/\bar{a})$ and transmission $(\mathrm{TC} = |\bar{T}|/\bar{a})$ coefficients are presented as functions of wave steepness in Figures 8 and 9.

The experimental and theoretical transmission coefficients correlate rather well. The correlation for the reflection coefficient can be improved by assuming an inertial coefficient greater than one.

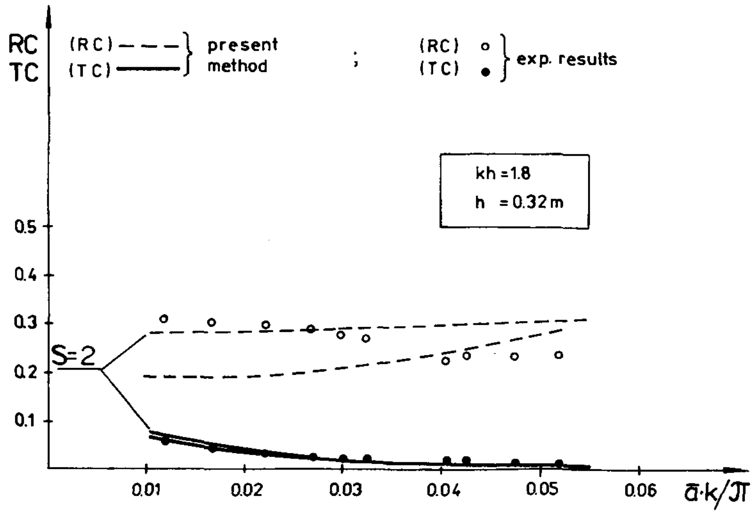

Figure 9. Reflection and transmission coefficients for trapezoidal layered breakwater (TW-2): dependence on wave steepness and on inertial coefficient ($S = 1$, $S = 2$)

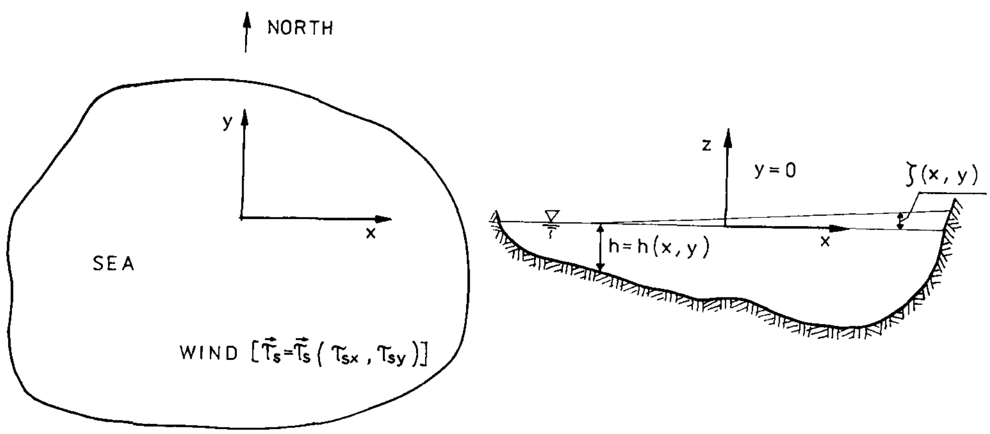

Figure 10. Definition sketch and co-ordinate system

This is indicated in Figure 9, where $S = 2$ as proposed by Le Méhauté.[19] A general discussion explaining the observed differences between the theoretical and experimental reflection coefficients has already been given by the author.[17,18]

Wind-driven currents in the sea

It is assumed that the fluid motion in the shallow sea (Figure 10) is governed by [20]

$$\frac{d\mathbf{V}}{dt} - 2\boldsymbol{\omega}_E \times \mathbf{V} + \frac{1}{\rho}\nabla_3(p + \gamma z) - k_v \frac{\partial^2 \mathbf{V}}{\partial z^2} - k_h \nabla^2 \mathbf{V} = 0, \tag{60}$$

$$\nabla_3 \cdot \mathbf{V} = 0, \tag{61}$$

where $\mathbf{V} = \mathbf{V}(u, v, w), \boldsymbol{\omega}_E$ is the Earth's angular frequency, $k_v(k_h)$ is the coefficient of exchange of momentum in the vertical (horizontal) direction, and ∇ and ∇_3 are two-dimensional and three-dimensional gradient operators.

It is further assumed that $k_h \nabla^2 \mathbf{V}, w \cong 0, p = p(\zeta)$, where ζ is the local water surface elevation. If the motion equations are integrated vertically, and if the boundary conditions at the free surface and the bottom are applied, then in a steady motion one obtains[20]

$$\nabla^2 \psi = \hat{p}(\ \ ), \tag{62}$$

where ψ is the stream function, and $\hat{p}(\ \ )$ is a function of ψ, ζ and h. Equation (62), with proper boundary conditions, can be solved iteratively using the method described earlier. In order to compare the numerical results with available analytical results a simple problem of wind-driven currents in a rectangular sea of constant depth has been considered.[21] Assuming that $\zeta \ll h$, $\tau_{sx} = -\tau_{11} \cos(\pi y/b_1), \tau_{sy} = 0$, the boundary-value problem for ψ_i (ith iteration) can be written as follows:

$$\hat{R}: \quad \nabla^2 \psi_i = c_{11} \sin \frac{\pi y}{b_1} - \alpha \frac{\partial \psi_{i-1}}{\partial x}, \tag{63}$$

$$\hat{S}: \quad \psi_i = 0, \tag{64}$$

where $\alpha = c_{12} \partial f_c / \partial y$; c_{11}, c_{12}, τ_{11} are constants and f_c is the Coriolis parameter.

The boundary-value problem, equations (63) and (64), has been solved for the case where f_c is a linear function of $y (\alpha = \pi/a_1, a_1 = b_1, N = 40, M = 200)$. The numerical results are in very good agreement with the analytical solution of Stommel[21] (Figure 11).

As another example it is assumed that only $w \cong 0$ and, as in the previous example, $p = p(\zeta)$. If the motion equation is integrated vertically and if the boundary conditions at the free surface and at the bottom are applied, then in a steady motion one obtains[22]

$$\nabla^4 \psi = \hat{q}(\ \ ) \tag{65}$$

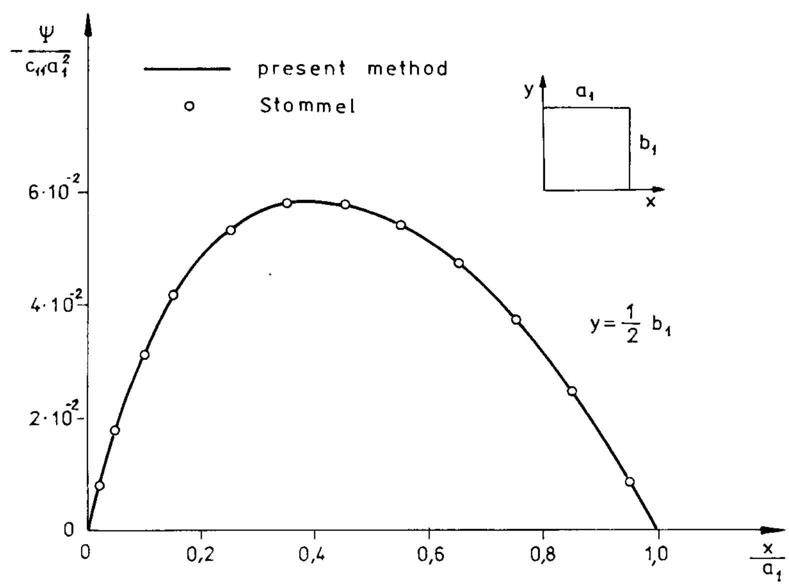

Figure 11. Stream function—comparison between numerical and analytical results

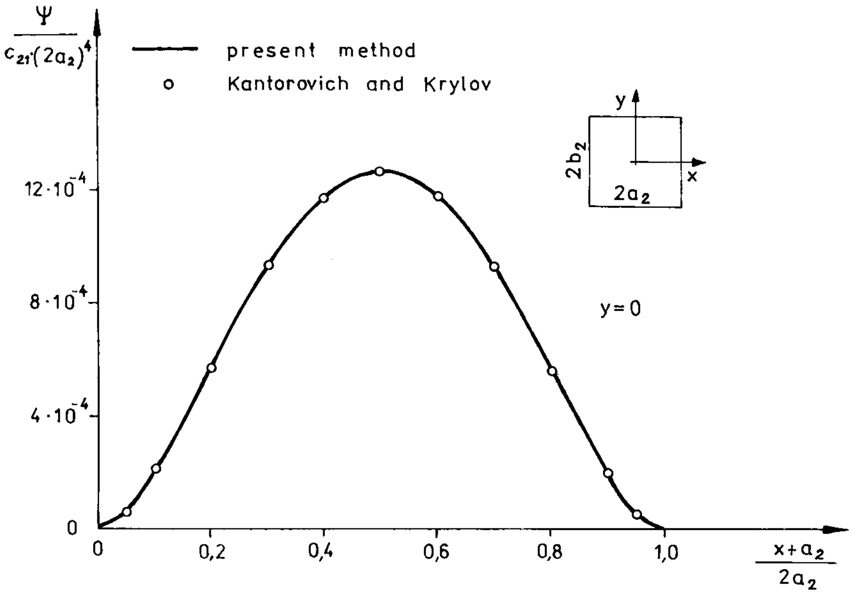

Figure 12. Stream function—comparison between numerical and analytical results

where $\hat{q}(\ )$ is a function of ψ, ζ and h.

The analytical solution of equation (65) with homogeneous boundary conditions for the stream function and its normal derivative is known only in some simple cases. Thus, in order to compare the numerical results with available analytical solutions further simplifications are necessary.

A sea of uniform depth and rectangular shape is to be considered. Assuming that $\zeta \ll h$, $\tau_{sx} = \tau_{21}y + \tau_{22}, \tau_{sy} = 0$, $f_c =$ constant, the boundary value problem for ψ can be written as follows:

$$\hat{R}: \quad \nabla^4\psi = c_{21}, \tag{66}$$

$$\hat{S}: \quad \psi = 0, \quad \frac{\partial\psi}{\partial n} = 0, \tag{67}$$

where τ_{21}, τ_{22} and c_{21} are constant. The boundary-value problem, equations (66) and (67), can be solved using well known particular solutions of equation (66). In the present approach the solutions have been obtained using the theory described earlier for solving the biharmonic non-homogeneous equation boundary-value problems. The numerical results ($a_2 = b_2, N = 40$, $M = 2$) are in very good agreement with the analytical solution of Kantorovich and Krylov[23] (Figure 12).

FURTHER DISCUSSION AND CONCLUSIONS

Integral equations arising from BEM formulation for plane fluid flow problems are usually solved under the assumption of uniformly distributed functions occurring in the integrands over the boundary or internal elements.[1,2,11,24] This procedure has been modified in this paper by using linearly varying functions and then applying simple local co-ordinate systems to the solution of the integral equations. The general advantages of the present BEM approach versus others approximate methods are consistent with those described by Banerjee and Butterfield[2] for conventional BEM. The presented approach, however, has some additional advantages. The main one is that fewer points are required for the discretization of the region and the boundary of the region than in the conventional BEM approach. Furthermore it is simple, it allows calculation of the tangential derivative of the seeking functions at the boundary elements, it can be used to define coefficients for numerical integration and it can be very helpful when the functions $\hat{p}$ or $\hat{q}$ are only defined pointwise, as is the case in many practical problems.

The present BEM approach has been shown to be an efficient technique in the numerical solution of common fluid flow problems encountered in ocean engineering. Since this method allows us to solve the boundary value problems for the Laplace or Poisson equations or for the homogeneous or non-homogeneous biharmonic equations, a wide range of fluid flow problems, including compressible or viscous ones, can be solved using this method.

The presented method also has applications other than fluid flow problems. Many problems of elastostatics, elastoplasticity, thin plates, electrostatics, diffusion and heat conduction can be solved efficiently using this method.

ACKNOWLEDGEMENTS

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APPENDIX I

The quantities $I^{\mathrm{III}}_{i,m}$, $I^{\mathrm{IV}}_{i,m}$ and $I^{\mathrm{V}}_{i,m}$ can be determined from the following formulae:

$$I^{\mathrm{III}}_{i,m} = I^{\mathrm{III},1}_{i,m}(c,d) - I^{\mathrm{III},1}_{i,m}(a,b), \tag{68}$$

$$I^{\mathrm{IV}}_{i,m} = I^{\mathrm{IV},1}_{i,m}(c,d) - I^{\mathrm{IV},1}_{i,m}(a,b), \tag{69}$$

$$I^{\mathrm{V}}_{i,m} = I^{\mathrm{V},1}_{i,m}(c,d) - I^{\mathrm{V},1}_{i,m}(a,b), \tag{70}$$

where

$$I^{\mathrm{III},1}_{i,m}(c,d) = A_{31}\tfrac{1}{4}\left\{\left[\tfrac{1}{3}(1+c^2)\lambda^3 + cd\lambda^2 + d^2\lambda + \frac{1}{3}\frac{3cd^3 + c^3d^3}{(1+c^2)^2}\right]E_1 + \frac{4}{3}\frac{d^3}{(1+c^2)^2}E_2 - \tfrac{5}{9}(1+c^2)\lambda^3 - \tfrac{5}{3}cd\lambda^2 - \frac{1}{3}\frac{5c^2d^2 + 7d^2}{1+c^2}\lambda\right\}\Bigg|_{\lambda_k}^{\lambda_l}, \tag{71}$$

$$I^{\mathrm{IV},1}_{i,m}(c,d) = A_{32}\frac{1}{2}\left\{\left[\tfrac{1}{3}c\lambda^3 + \tfrac{1}{2}d\lambda^2 + \frac{1}{6}\frac{d^3 - c^2d^3}{(1+c^2)^2}\right]E_1 - \frac{2}{3}\frac{cd^3}{(1+c^2)^2}E_2 + \tfrac{2}{3}E_3\lambda^3 - \tfrac{8}{9}c\lambda^3 - \tfrac{7}{6}d\lambda^2 + \frac{1}{3}\frac{cd^2}{1+c^2}\lambda\right\}\Bigg|_{\lambda_k}^{\lambda_l}, \tag{72}$$

$$I^{\mathrm{V},1}_{i,m}(c,d) = A_{33}\frac{1}{2}\left[\left(\tfrac{1}{2}c\lambda^2 + d\lambda + \frac{1}{2}\frac{cd^2}{1+c^2}\right)E_1 + \frac{d^2}{1+c^2}E_2 + E_3\lambda^2 - \tfrac{3}{2}c\lambda^2 - 3d\lambda\right]\Bigg|_{\lambda_k}^{\lambda_l} \tag{73}$$

and

$$E_1(c,d) = \ln[(1+c^2)\lambda^2 + 2cd\lambda + d^2], \tag{74}$$

$$E_2(c,d) = \arctan\left[\frac{(1+c^2)\lambda + cd}{d}\right], \tag{75}$$

$$E_3(c,d) = \arctan\left[\frac{c\lambda + d}{\lambda}\right]. \tag{76}$$

APPENDIX II

The quantities $\mathrm{II}^{\mathrm{V}}_{i,m}$, $\mathrm{II}^{\mathrm{VI}}_{i,m}$ and $\mathrm{II}^{\mathrm{VII}}_{i,m}$ can be determined from the following formulae:

$$\mathrm{II}^{\mathrm{V}}_{i,m} = \mathrm{II}^{\mathrm{V},1}_{i,m}(c,d) - \mathrm{II}^{\mathrm{V},1}_{i,m}(a,b), \tag{77}$$

$$\mathrm{II}^{\mathrm{VI}}_{i,m} = \mathrm{II}^{\mathrm{VI},1}_{i,m}(c,d) - \mathrm{II}^{\mathrm{VI},1}_{i,m}(a,b), \tag{78}$$

$$\mathrm{II}^{\mathrm{VII}}_{i,m} = \mathrm{II}^{\mathrm{VII},1}_{i,m}(c,d) - \mathrm{II}^{\mathrm{VII},1}_{i,m}(a,b), \tag{79}$$

$$\mathrm{II}^{\mathrm{V},1}_{i,m}(c,d) = B_{51}\frac{1}{1+c^2}[\tfrac{1}{8}T_{00}(1+c^2)(E_1 - \tfrac{1}{2}) - \tfrac{1}{5}T_1\lambda^5 - \tfrac{1}{4}T_7\lambda^4 - \tfrac{1}{3}T_9\lambda^3$$
$$- \tfrac{1}{2}T_{11}\lambda^2 - T_{13}\lambda - \tfrac{1}{2}T_{15}E_1 + (T_{15}c + T_{13}d)\,E_2]|^{\lambda_l}_{\lambda_k}, \tag{80}$$

where

$$T_{00}(c,d) = \tfrac{1}{5}(1+c^2)^2\lambda^5 + (1+c^2)cd\lambda^4 + [\tfrac{2}{3}(1+c^2)d^2 + \tfrac{4}{3}c^2d^2]\lambda^3 + 2cd^3\lambda^2 + d^4\lambda, \tag{81}$$

$$T_1(c,d) = \tfrac{1}{20}(1+c^2)^3, \tag{82}$$

$$T_2(c,d) = \tfrac{3}{10}(1+c^2)^2cd, \tag{83}$$

$$T_3(c,d) = \tfrac{1}{6}(1+c^2)^2d^2 + \tfrac{7}{12}(1+c^2)c^2d^2, \tag{84}$$

$$T_4(c,d) = \tfrac{2}{3}(1+c^2)cd^3 + \tfrac{1}{3}c^3d^3, \tag{85}$$

$$T_5(c,d) = \tfrac{1}{4}(1+c^2)d^4 + \tfrac{1}{2}c^2d^4, \tag{86}$$

$$T_6(c,d) = \tfrac{1}{4}cd^5, \tag{87}$$

$$T_7(c,d) = -2T_1cd/(1+c^2) + T_2, \tag{88}$$

$$T_8(c,d) = -T_1d^2/(1+c^2) + T_3, \tag{89}$$

$$T_9(c,d) = -2T_7cd/(1+c^2) + T_8, \tag{90}$$

$$T_{10}(c,d) = -T_7d^2/(1+c^2) + T_4, \tag{91}$$

$$T_{11}(c,d) = -2T_9cd/(1+c^2) + T_{10}, \tag{92}$$

$$T_{12}(c,d) = -T_9d^2/(1+c^2) + T_5, \tag{93}$$

$$T_{13}(c,d) = -2T_{11}cd/(1+c^2) + T_{12}, \tag{94}$$

$$T_{14}(c,d) = -T_{11}d^2/(1+c^2) + T_6, \tag{95}$$

$$T_{15}(c,d) = -2T_{13}cd/(1+c^2) + T_{14}, \tag{96}$$

$$\mathrm{II}_{i,m}^{\mathrm{VI},1}(c,d)=B_{52}\frac{1}{1+c^2}[U_{00}(1+c^2)E_1+U_{01}(1+c^2)-\tfrac{1}{5}U_1\lambda^5-\tfrac{1}{4}U_6\lambda^4-\tfrac{1}{3}U_8\lambda^3$$
$$-\tfrac{1}{2}U_{10}\lambda^2-U_{12}\lambda-\tfrac{1}{2}U_{14}E_1+(U_{14}c+U_{12}d)\,E_2]|_{\lambda_k}^{\lambda_l} \tag{97}$$

where

$$U_{00}(c,d)=\tfrac{1}{5}(\tfrac{1}{2}c+\tfrac{1}{6}c^3)\lambda^5+\tfrac{1}{4}(\tfrac{1}{2}d+\tfrac{1}{2}c^2d)\lambda^4+\tfrac{1}{6}cd^2\lambda^3+\tfrac{1}{12}d^3\lambda^2, \tag{98}$$
$$U_{01}(c,d)=-\tfrac{1}{5}(\tfrac{2}{3}c+\tfrac{1}{9}c^3-\tfrac{2}{3}E_3)\lambda^5-\tfrac{1}{4}(\tfrac{2}{3}d+\tfrac{1}{3}c^2d)\lambda^4-\tfrac{1}{9}cd^2\lambda^3-\tfrac{1}{18}d^3\lambda^2, \tag{99}$$
$$U_1(c,d)=\tfrac{1}{5}(1+c^2)c+\tfrac{1}{15}(1+c^2)c^3, \tag{100}$$
$$U_2(c,d)=\tfrac{1}{4}(1+c^2)^2d+\tfrac{1}{5}c^2d+\tfrac{1}{15}c^4d-\tfrac{2}{15}d, \tag{101}$$
$$U_3(c,d)=\tfrac{7}{12}(1+c^2)cd^2, \tag{102}$$
$$U_4(c,d)=\tfrac{1}{6}(1+c^2)d^3+\tfrac{1}{3}c^2d^3, \tag{103}$$
$$U_5(c,d)=\tfrac{1}{6}cd^4, \tag{104}$$
$$U_6(c,d)=-2U_1cd/(1+c^2)+U_2, \tag{105}$$
$$U_7(c,d)=-U_1d^2/(1+c^2)+U_3, \tag{106}$$
$$U_8(c,d)=-2U_6cd/(1+c^2)+U_7, \tag{107}$$
$$U_9(c,d)=-U_6d^2/(1+c^2)+U_4, \tag{108}$$
$$U_{10}(c,d)=-2U_8cd/(1+c^2)+U_9, \tag{109}$$
$$U_{11}(c,d)=-U_8d^2/(1+c^2)+U_5, \tag{110}$$
$$U_{12}(c,d)=-2U_{10}cd/(1+c^2)+U_{11}, \tag{111}$$
$$U_{13}(c,d)=-U_{10}d^2/(1+c^2), \tag{112}$$
$$U_{14}(c,d)=-2U_{12}cd/(1+c^2)+U_{13}, \tag{113}$$

and

$$\mathrm{II}_{i,m}^{\mathrm{VII},1}(c,d)=B_{53}\frac{1}{1+c^2}[W_{00}(1+c^2)E_1+W_{01}(1+c^2)-\tfrac{1}{4}W_1\lambda^4-\tfrac{1}{3}W_6\lambda^3-\tfrac{1}{2}W_8\lambda^2$$
$$-W_{10}\lambda-\tfrac{1}{2}W_{12}E_1+(W_{12}c+W_{10}d)E_2]|_{\lambda_k}^{\lambda_l}, \tag{114}$$

where

$$W_{00}(c,d)=\tfrac{1}{4}(\tfrac{1}{2}c+\tfrac{1}{6}c^3)\lambda^4+\tfrac{1}{6}(1+c^2)d\lambda^3+\tfrac{1}{4}cd^2\lambda^2+\tfrac{1}{6}d^3\lambda, \tag{115}$$
$$W_{01}(c,d)=-\tfrac{1}{4}(\tfrac{2}{3}c+\tfrac{1}{9}c^3-\tfrac{2}{3}E_3)\lambda^4-\tfrac{1}{3}(\tfrac{2}{3}d+\tfrac{1}{3}c^2d)\lambda^3-\tfrac{1}{6}cd^2\lambda^2-\tfrac{1}{9}d^3\lambda, \tag{116}$$
$$W_1(c,d)=\tfrac{1}{4}(1+c^2)c+\tfrac{1}{12}(1+c^2)c^3, \tag{117}$$
$$W_2(c,d)=\tfrac{1}{3}(1+c^2)^2d+\tfrac{1}{4}c^2d+\tfrac{1}{12}c^4d-\tfrac{1}{6}d \tag{118}$$
$$W_3(c,d)=\tfrac{5}{6}(1+c^2)cd^2, \tag{119}$$
$$W_4(c,d)=\tfrac{1}{3}(1+c^2)d^3+\tfrac{1}{2}c^2d^3, \tag{120}$$
$$W_5(c,d)=\tfrac{1}{3}cd^4, \tag{121}$$
$$W_6(c,d)=-2W_1cd/(1+c^2)+W_2, \tag{122}$$

$$W_7(c,d) = -W_1 d^2/(1+c^2) + W_3, \tag{123}$$

$$W_8(c,d) = -2W_6 cd/(1+c^2) + W_7, \tag{124}$$

$$W_9(c,d) = -W_6 d^2/(1+c^2) + W_4, \tag{125}$$

$$W_{10}(c,d) = -2W_8 cd/(1+c^2) + W_9, \tag{126}$$

$$W_{11}(c,d) = -W_8 d^2/(1+c^2) + W_5, \tag{127}$$

$$W_{12}(c,d) = -2W_{10} cd/(1+c^2) + W_{11} \tag{128}$$

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